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A Lundberg-type inequality for an inhomogeneous renewal risk model

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Abstract We obtain a Lundberg-type inequality in the case of an inhomogeneous renewal risk model. We consider the model with independent, but not necessarily identically distributed, claim sizes and the interoccurrence times. In order to prove the main theorem, we first formulate and prove an auxiliary lemma on large values of a sum of random variables asymptotically drifted in the negative direction.

Keywords Inhomogeneous model, renewal model, Lundberg-type inequality, exponential bound, ruin probability

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1 Introduction

The classical risk model and the renewal risk model are two models that are traditionally used to describe the nonlife insurance business. The classical risk model was introduced by Lundberg and Cramér about a century ago (see [8, 14, 15] for the source papers and [18] for the historical environment). In this risk model, it is assumed that interarrival times are identically distributed, exponential, and independent

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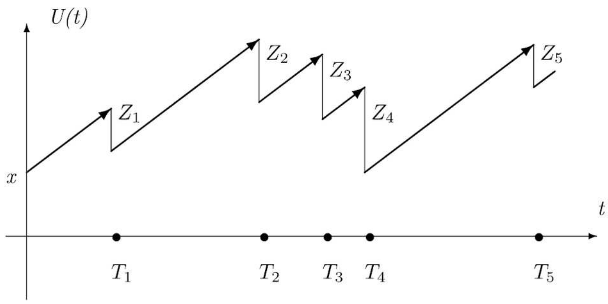

Fig. 1. Behavior of the surplus process.

random variables. In 1957, the Danish mathematician E. Sparre Andersen proposed the renewal risk model to describe the surplus process of the insurance company. In the renewal risk model, the claim sizes and the interarrival times are independent, identically distributed, nonnegative random variables (see [2] for the source paper and [22] for additional details). In this paper, we assume that interoccurrence times and claim sizes are nonnegative random variables (r.v.s) that are not necessarily identically distributed. We call such a model the inhomogeneous model and present its exact definition. It is evident that the inhomogeneous renewal risk model reflects better the real insurance activities in comparison with the classical risk model or with the renewal (homogeneous) risk model.

Definition 1. We say that the insurer's surplus $U(t)$ varies according to the inhomogeneous renewal risk model if

$$U(t) = U(\omega, t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i$$

for all $t \geqslant 0$. Here:

- $x \geqslant 0$ is the initial reserve;
- claim sizes $\{Z_1, Z_2, \ldots\}$ form a sequence of independent (not necessarily identically distributed) nonnegative r.v.s;
- $c > 0$ is the constant premium rate;
- $\Theta(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}} = \sup\{n \geqslant 0 : T_n \leqslant t\}$ is the number of claims in the interval $[0, t]$, where $T_0 = 0$, $T_n = \theta_1 + \theta_2 + \cdots + \theta_n$, $n \geqslant 1$, and the interarrival times $\{\theta_1, \theta_2, \ldots\}$ are independent (not necessarily identically distributed), nonnegative, and nondegenerate at zero r.v.s;
- the sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent.

A typical path of the surplus process of an insurance company is shown in Fig. 1.

If all claim sizes $\{Z_1, Z_2, \ldots\}$ and all interarrival times $\{\theta_1, \theta_2, \ldots\}$ are identically distributed, then the inhomogeneous renewal risk model becomes the homogeneous renewal risk model.

The *time of ruin* and *the ruin probability* are the main critical characteristics of any risk model. Let $\mathcal{B}$ denote the event of ruin. We suppose that

$$\mathcal{B} = \bigcup_{t\geqslant 0}\{\omega : U(\omega,t) < 0\} = \bigcup_{t\geqslant 0}\Biggl\{\omega : x + ct - \sum_{i=1}^{\Theta(t)} Z_i < 0\Biggr\},$$

that is, that ruin occurs if at some time $t \geqslant 0$ the surplus of the insurance company becomes negative or, in other words, the insurer becomes unable to pay all the claims. The first time τ when the surplus drops to a level less than zero is called *the time of ruin*, that is, τ is the extended r.v. for which

$$\tau = \tau(\omega) = \begin{cases} \inf\{t \geqslant 0 : U(\omega,t) < 0\} & \text{if } \omega \in \mathcal{B}, \\ \infty & \text{if } \omega \notin \mathcal{B}. \end{cases}$$

The ruin probability ψ is defined by the equality

$$\psi(x) = \mathbb{P}(\mathcal{B}) = \mathbb{P}(\tau = \infty).$$

Usually, we suppose that the main parameter of the ruin probability is the initial reserve x, though actually the ruin probability, together with time of ruin, depends on all components of the renewal risk model.

All trajectories of the process $U(t)$ are increasing functions between times T_n and T_{n+1} for all $n = 0, 1, 2, \dots$. Therefore, the random variables $U(\theta_1 + \theta_2 + \cdots + \theta_n)$, $n \geqslant 1$, are the local minimums of the trajectories. Consequently, we can express the ruin probability in the following way (for details, see [9] or [16]):

$$\begin{aligned}
\psi(x) &= \mathbb{P}\Bigl(\inf_{n\in\mathbb{N}} U(\theta_1 + \theta_2 + \cdots + \theta_n) < 0\Bigr) \\
&= \mathbb{P}\Biggl(\inf_{n\in\mathbb{N}}\Biggl\{x + c(\theta_1 + \theta_2 + \cdots + \theta_n) - \sum_{i=1}^{\Theta(\theta_1+\cdots+\theta_n)} Z_i\Biggr\} < 0\Biggr) \\
&= \mathbb{P}\Biggl(\inf_{n\in\mathbb{N}}\Biggl\{x - \sum_{i=1}^{n}(Z_i - c\theta_i)\Biggr\} < 0\Biggr) \\
&= \mathbb{P}\Biggl(\sup_{n\in\mathbb{N}}\Biggl\{\sum_{i=1}^{n}(Z_i - c\theta_i)\Biggr\} > x\Biggr).
\end{aligned}$$

Further, in this paper, we restrict our study to the so-called *Lundberg-type inequality*. An exponential bound for the ruin probability is usually called a Lundberg-type inequality. We further give the well-known exponential bound for $\psi(x)$ in homogeneous renewal risk model (see, for instance, Chapters "Lundberg Inequality for Ruin Probability", "Collective Risk Theory", "Adjustment Coefficient," or "Cramer–Lundberg Asymptotics" in [21]).

Theorem 1. *Let the net profit condition* $\mathbb{E}Z_1 - c\mathbb{E}\theta_1 < 0$ *hold, and let* $\mathbb{E}\mathrm{e}^{hZ_1} < \infty$ *for some* $h > 0$ *in the homogeneous renewal risk model. Then, there is a number* $H > 0$ *such that*

$$\psi(x) \leqslant \mathrm{e}^{-Hx} \tag{1}$$

for all $x \geqslant 0$. *If* $\mathbb{E}\mathrm{e}^{R(Z_1 - c\theta_1)} = 1$ *for some positive* R, *then we can take* $H = R$ *in* (1).

There exist a lot of different proofs of this theorem. The main ways to prove inequality (1) are described in Chapter "Lundberg Inequality for Ruin Probability" of [21]. Details of some existing proofs were given, for instance, by Asmussen and Albrecher [3], Embrechts, Klüppelberg, and Mikosch [9], Embrechts and Veraverbeke [10], Gerber [11], and Mikosch [16]. We note only that the bound (1) can be proved using the exponential tail bound of Sgibnev [19] and the inequality $\psi(0) < 1$.

The following theorem is the main statement of the paper.

Theorem 2. *Let the claim sizes* $\{Z_1, Z_2, \dots\}$ *and the interarrival times* $\{\theta_1, \theta_2, \dots\}$ *form an inhomogeneous renewal risk model described in Definition 1. Further, let the following three conditions be satisfied:*

$$(\mathcal{C}1)\ \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\gamma Z_i} < \infty \quad \textit{for some } \gamma > 0,$$

$$(\mathcal{C}2)\ \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\big(\theta_i \mathbb{1}_{\{\theta_i > u\}}\big) = 0,$$

$$(\mathcal{C}3)\ \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\,\mathbb{E}Z_i - c\mathbb{E}\theta_i) < 0.$$

Then, there are constants $c_1 > 0$ *and* $c_2 \geqslant 0$ *such that* $\psi(x) \leqslant \mathrm{e}^{-c_1 x}$ *for all* $x \geqslant c_2$.

The inhomogeneous renewal risk model differs from the homogeneous one because the independence and/or homogeneous distribution of sequences of random variables $\{Z_1, Z_2, \dots\}$ and/or $\{\theta_1, \theta_2, \dots\}$ are no longer required. The changes depend on how the inhomogeneity in a particular model is understood. In Definition 1, we have chosen one of two possible directions used in numerous articles that deal with inhomogeneous renewal risk models. This is due to the fact that an inhomogeneity can be considered as the possibility to have either differently distributed or dependent r.v.s in the sequences.

The possibility to have differently distributed r.v.s was considered, for instance, in [5, 6, 12, 17]. In the first three works, the discrete-time inhomogeneous risk model was considered. In such a model, the interarrival times are fixed, and the claims $\{Z_1, Z_2, \dots\}$ are independent, not necessary identically distributed, integer valued r.v.s. In [17], the authors consider the model where the interarrival times are identically distributed and have a particular distribution, whereas the claims are differently distributed with distributions belonging to a particular class. Bernackaitė and Šiaulys [4] deal with an inhomogeneous renewal risk model where the r.v.s $\{\theta_1, \theta_2, \dots\}$ are not necessarily identically distributed, but the claim sizes $\{Z_1, Z_2, \dots\}$ have a common distribution function. In this article, we consider a more general renewal risk model. In the main theorem, we assume that not only r.v.s $\{\theta_1, \theta_2, \dots\}$ are not necessarily identically distributed, but also the same holds for the sequence of claim sizes $\{Z_1, Z_2, \dots\}$.

There is another approach to inhomogeneous renewal risk models, which implies the possibility to have dependence in sequences and mainly found in works by Chinese researchers. In this kind of models, the sequences $\{Z_1, Z_2, \dots\}$ and $\{\theta_1, \theta_2, \dots\}$ consist of identically distributed r.v.s, but there may be some kind of dependence between them. Results for such models can be found, for instance, in [7] and [23]. Another interpretation of dependence is also possible, where r.v.s in both

sequences $\{Z_1, Z_2, \dots\}$ and $\{\theta_1, \theta_2, \dots\}$ still remain independent. Instead of that, the mutual independence of these two sequences is no longer required. The idea of this kind of dependence belongs to Albrecher and Teugels [1], and this encouraged Li, Tang, and Wu [13] to study renewal risk models having this dependence structure.

The rest of the paper consists of two sections. In Section 2, we formulate and prove an auxiliary lemma. The proof of the main theorem is presented in Section 3.

2 Auxiliary lemma

In order to prove the main theorem, we need an auxiliary lemma. In Lemma 1, the conditions for r.v.s $\eta_1, \eta_2, \eta_3, \dots$ are taken from articles by Smith [20] and Bernackaitė and Šiaulys [4].

Lemma 1. *Let $\eta_1, \eta_2, \eta_3, \dots$ be independent r.v.s such that*

$$\begin{aligned}
&(\mathcal{C}1^*)\ \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i} < \infty \quad \textit{for some } \delta > 0,\\
&(\mathcal{C}2^*)\ \lim_{u\to\infty}\sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -u\}}\big) = 0,\\
&(\mathcal{C}3^*)\ \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\eta_i < 0.
\end{aligned}$$

Then, there are constants $c_3 > 0$ and $c_4 > 0$ such that

$$\mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\right) \leqslant c_3\mathrm{e}^{-c_4x}$$

for all $x \geqslant 0$.

Proof. First, we observe that, for all $x \geqslant 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\right) &= \mathbb{P}\left(\bigcup_{k=1}^{\infty}\left\{\sum_{i=1}^{k}\eta_i > x\right\}\right)\\
&\leqslant \sum_{k=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right). \qquad (2)
\end{aligned}$$

By Chebyshev's inequality, for all $x \geqslant 0$, $0 < y \leqslant \delta$, and $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right) &= \mathbb{P}\big(\mathrm{e}^{y\sum_{i=1}^{k}\eta_i} > \mathrm{e}^{yx}\big)\\
&\leqslant \mathrm{e}^{-yx}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y\eta_i}. \qquad (3)
\end{aligned}$$

Moreover, for all $i \in \mathbb{N}$, $0 < y \leqslant \delta$, and $u > 0$, we have

$$\mathbb{E}\mathrm{e}^{y\eta_i} = 1 + y\mathbb{E}\eta_i + \mathbb{E}\big(\mathrm{e}^{y\eta_i} - 1 - y\eta_i\big) \qquad (4)$$

and

$$\begin{aligned}
&\mathbb{E}\big(\mathrm{e}^{y\eta_i}-1-y\eta_i\big)\\
&\quad=\mathbb{E}\big(\big(\mathrm{e}^{y\eta_i}-1\big)\mathbb{1}_{\{\eta_i\leqslant -u\}}\big)-y\mathbb{E}\big(\eta_i\mathbb{1}_{\{\eta_i\leqslant -u\}}\big)\\
&\qquad+\mathbb{E}\big(\big(\mathrm{e}^{y\eta_i}-1-y\eta_i\big)\mathbb{1}_{\{-u<\eta_i\leqslant 0\}}\big)+\mathbb{E}\big(\big(\mathrm{e}^{y\eta_i}-1-y\eta_i\big)\mathbb{1}_{\{\eta_i>0\}}\big).
\end{aligned}$$

In order to evaluate the absolute value of the remainder term in (4), we need the following inequalities:

$$\begin{aligned}
\big|\mathrm{e}^{v}-1\big|&\leqslant |v|,\quad v\leqslant 0,\\
\big|\mathrm{e}^{v}-v-1\big|&\leqslant \frac{v^2}{2},\quad v\leqslant 0,\\
\big|\mathrm{e}^{v}-v-1\big|&\leqslant \frac{v^2}{2}\mathrm{e}^{v},\quad v\geqslant 0.
\end{aligned}$$

Using them, we get

$$\begin{aligned}
&\big|\mathbb{E}\big(\mathrm{e}^{y\eta_i}-1-y\eta_i\big)\big|\\
&\quad\leqslant 2y\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -u\}}\big)+\frac{y^2}{2}\mathbb{E}\big(\eta_i^2\mathbb{1}_{\{-u<\eta_i\leqslant 0\}}\big)+\frac{y^2}{2}\mathbb{E}\big(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{\eta_i>0\}}\big)\\
&\quad\leqslant 2y\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -u\}}\big)+\frac{y^2u^2}{2}+\frac{y^2}{2}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{\eta_i>0\}}\big),
\end{aligned}\tag{5}$$

where $i\in\mathbb{N}$, $0<y\leqslant\delta$, and $u>0$.

Since

$$\lim_{v\to\infty}\frac{\mathrm{e}^{\delta v/2}}{v^2}=\infty,$$

we have

$$\mathrm{e}^{\delta v/2}\geqslant v^2$$

for all $v\geqslant c_5$, where $c_5=c_5(\delta)>0$.

Therefore,

$$\begin{aligned}
&\sup_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i^2\mathrm{e}^{\delta\eta_i/2}\mathbb{1}_{\{\eta_i>0\}}\big)\\
&\quad\leqslant \sup_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i^2\mathrm{e}^{\delta\eta_i/2}\mathbb{1}_{\{0<\eta_i\leqslant c_5\}}\big)+\sup_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i^2\mathrm{e}^{\delta\eta_i/2}\mathbb{1}_{\{\eta_i>c_5\}}\big)\\
&\quad\leqslant \big(c_5^2+1\big)\sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\delta\eta_i}<\infty.
\end{aligned}\tag{6}$$

Choosing $u=\frac{1}{\sqrt[4]{y}}$ in (5) and using (6), we get

$$\begin{aligned}
&\big|\mathbb{E}\big(\mathrm{e}^{y\eta_i}-1-y\eta_i\big)\big|\\
&\quad\leqslant 2y\sup_{i\in\mathbb{N}}\mathbb{E}\Big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -\frac{1}{\sqrt[4]{y}}\}}\Big)+\frac{y^{\frac{3}{2}}}{2}+\frac{y^2}{2}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{\eta_i>0\}}\big)
\end{aligned}$$

$$\leqslant y\left(2\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -\frac{1}{\sqrt[4]{y}}\}}\big)+\frac{y^{\frac{1}{2}}}{2}+\frac{y}{2}\big(c_5^2+1\big)\sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\delta\eta_i}\right)$$
$$=: y\alpha(y), \tag{7}$$

where $i\in\mathbb{N}$, $y\in(0,\delta/2]$, $c_5=c_5(\delta)$, and

$$\alpha(y)=2\sup_{i\in\mathbb{N}}\mathbb{E}\Big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -\frac{1}{\sqrt[4]{y}}\}}\Big)+\frac{y^{\frac{1}{2}}}{2}+\frac{y}{2}\big(c_5^2+1\big)\sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\delta\eta_i}.$$

Conditions ($\mathcal{C}1^*$) and ($\mathcal{C}2^*$) imply that $\alpha(y)\downarrow 0$ as $y\to 0$.

For an arbitrary positive v, we have

$$\begin{aligned}\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i<0\}}\big)&=\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{-v<\eta_i<0\}}+|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -v\}}\big)\\&\leqslant v+\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -v\}}\big).\end{aligned}$$

So, condition ($\mathcal{C}2^*$) implies that

$$\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i<0\}}\big)<\infty. \tag{8}$$

Denote

$$\widehat{y}=\min\Big\{\delta/2,1/\Big(2\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i<0\}}\big)\Big)\Big\}.$$

If $y\in(0,\widehat{y}]$, then

$$\begin{aligned}y\big(\mathbb{E}\eta_i+\alpha(y)\big)&>y\mathbb{E}\eta_i\\&=y\mathbb{E}\big(\eta_i\mathbb{1}_{\{\eta_i\geqslant 0\}}+\eta_i\mathbb{1}_{\{\eta_i<0\}}\big)\\&\geqslant y\mathbb{E}\big(\eta_i\mathbb{1}_{\{\eta_i<0\}}\big)\\&\geqslant \widehat{y}\inf_{i\in\mathbb{N}}\mathbb{E}\big(\eta_i\mathbb{1}_{\{\eta_i<0\}}\big)\\&=-\widehat{y}\sup_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i<0\}}\big)\\&\geqslant -1/2\end{aligned}$$

for all $i\in\mathbb{N}$.

Therefore, (3), (4), (7), and the well-known inequality

$$\ln(1+u)\leqslant u,\quad u>-1,$$

imply that

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right) &\leqslant \mathrm{e}^{-yx}\prod_{i=1}^{k}\bigl(1+y\mathbb{E}\eta_i+\mathbb{E}\bigl(\mathrm{e}^{y\eta_i}-1-y\eta_i\bigr)\bigr)\\ &\leqslant \mathrm{e}^{-yx}\prod_{i=1}^{k}\bigl(1+y\bigl(\mathbb{E}\eta_i+\alpha(y)\bigr)\bigr)\\ &= \exp\left\{-yx+\sum_{i=1}^{k}\ln\bigl(1+y\bigl(\mathbb{E}\eta_i+\alpha(y)\bigr)\bigr)\right\}\\ &\leqslant \exp\left\{-yx+y\sum_{i=1}^{k}\mathbb{E}\eta_i+yk\alpha(y)\right\}, \end{aligned}\tag{9}$$

where $k \in \mathbb{N}$, $x \geqslant 0$, and $y \in (0, \widehat{y}\,]$.

By estimate (8) and condition ($\mathcal{C}3^*$) we can suppose that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\eta_i = -c_6$$

for some positive constant c_6. Then we have

$$\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\eta_i \leqslant -\frac{c_6}{2}$$

for $k \geqslant M+1$ with some $M \geqslant 1$. Moreover, there exists $y^* \in (0, \widehat{y}\,]$ such that $\alpha(y^*) \leqslant c_6/4$ since $\alpha(y) \downarrow 0$ as $y \to 0$.

Using results from (2), (3), and (9), we derive

$$\begin{aligned}&\mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\right)\\ &\quad\leqslant \sum_{k=1}^{M}\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right)+\sum_{k=M+1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right)\\ &\quad\leqslant \sum_{k=1}^{M}\mathrm{e}^{-y^*x}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y^*\eta_i}+\sum_{k=M+1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{k}\eta_i > x\right)\\ &\quad\leqslant \sum_{k=1}^{M}\mathrm{e}^{-y^*x}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y^*\eta_i}+\sum_{k=M+1}^{\infty}\mathrm{e}^{-y^*x+y^*\sum_{i=1}^{k}\mathbb{E}\eta_i+y^*k\alpha(y^*)}\\ &\quad\leqslant \mathrm{e}^{-y^*x}\left(\sum_{k=1}^{M}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y^*\eta_i}+\sum_{k=0}^{\infty}\mathrm{e}^{-ky^*c_6/4}\right)\\ &\quad\leqslant \mathrm{e}^{-y^*x}\left(\sum_{k=1}^{M}\prod_{i=1}^{k}\varDelta+\frac{1}{1-\mathrm{e}^{-y^*c_6/4}}\right)\\ &\quad= \mathrm{e}^{-y^*x}\left(\frac{\varDelta(\varDelta^M-1)}{\varDelta-1}+\frac{\mathrm{e}^{y^*c_6/4}}{\mathrm{e}^{y^*c_6/4}-1}\right) =: c_3\mathrm{e}^{-c_4x}, \end{aligned}$$

where

$$\begin{aligned} &x \geqslant 0, \\ &\varDelta = 1 + \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i}, \\ &c_3 = \frac{\varDelta(\varDelta^M - 1)}{\varDelta - 1} + \frac{e^{y^* c_6/4}}{e^{y^* c_6/4} - 1}, \\ &c_4 = y^* \in (0, \widehat{y}] \end{aligned}$$

with $M \geqslant 1$, $c_6 > 0$, and $\widehat{y} > 0$ defined previously. The lemma is now proved. □

3 Proof of Theorem 2

In this section, we prove Theorem 2.

Proof. Since

$$\psi(x) = \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{i=1}^{n}(Z_i - c\theta_i)\right\} > x\right),$$

the desired bound of Theorem 2 can be derived from auxiliary Lemma 1.

Namely, supposing that r.v.s $Z_i - c\theta_i$, $i \in \{1, 2, \ldots\}$, satisfy all conditions of Lemma 1, we get

$$\psi(x) \leqslant c_7 \mathrm{e}^{-c_8 x}$$

for all $x \geqslant 0$ with some positive c_7, c_8 independent of x.

Therefore,

$$\psi(x) \leqslant c_7 \mathrm{e}^{-c_8 x/2} \mathrm{e}^{-c_8 x/2} \leqslant \mathrm{e}^{-c_8 x/2},$$

with $x \geqslant \max\{0, (2\ln c_7)/c_8\}$,

Thus, it suffices to check all three assumptions in our lemma with random variables $Z_i - c\theta_i$, $i \in \mathbb{N}$. The requirement $(\mathcal{C}3^*)$ of Lemma 1 is evidently satisfied by condition $(\mathcal{C}3)$.

Next, it follows from $(\mathcal{C}1)$ that

$$\sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\gamma(Z_i - c\theta_i)} \leqslant \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\gamma Z_i} < \infty.$$

So, the requirement $(\mathcal{C}1^*)$ holds too.

It remains to prove that

$$\lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|Z_i - c\theta_i| \mathbb{1}_{\{Z_i - c\theta_i \leqslant -u\}}\big) = 0. \tag{10}$$

To establish this, we use the inequality

$$\begin{aligned} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|Z_i - c\theta_i| \mathbb{1}_{\{Z_i - c\theta_i \leqslant -u\}}\big) &\leqslant \sup_{i\in\mathbb{N}} \mathbb{E}\big(Z_i \mathbb{1}_{\{Z_i - c\theta_i \leqslant -u\}}\big) \\ &\quad + c \sup_{i\in\mathbb{N}} \mathbb{E}\big(\theta_i \mathbb{1}_{\{Z_i - c\theta_i \leqslant -u\}}\big). \end{aligned} \tag{11}$$

Taking the limit as $u \to \infty$ in the first summand of the right side of inequality (11), we get

$$\begin{aligned}
&\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big)\\
&\quad\leqslant \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i\leqslant \frac{u}{2c}\}}\big)\\
&\qquad+\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\
&\quad\leqslant \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i\leqslant -u/2\}}\big)\\
&\qquad+\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\
&\quad= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\
&\quad\leqslant \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\
&\quad= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}Z_i\mathbb{P}\Big(\theta_i>\frac{u}{2c}\Big)\\
&\quad\leqslant \sup_{i\in\mathbb{N}}\mathbb{E}Z_i\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{P}\Big(\theta_i>\frac{u}{2c}\Big).
\end{aligned}\tag{12}$$

Since $x \leqslant \mathrm{e}^{\gamma x}/\gamma$, $x \geqslant 0$, condition $(\mathcal{C}1)$ implies that

$$\sup_{i\in\mathbb{N}}\mathbb{E}Z_i<\infty.\tag{13}$$

In addition,

$$\begin{aligned}
\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{P}\Big(\theta_i>\frac{u}{2c}\Big) &= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\left(\frac{\theta_i\mathbb{1}_{\{\theta_i>\frac{u}{2c}\}}}{\theta_i}\right)\\
&\leqslant \lim_{u\to\infty}\frac{2c}{u}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i>\frac{u}{2c}\}}\big)=0
\end{aligned}\tag{14}$$

by condition $(\mathcal{C}2)$.

Therefore, relations (12), (13), and (14) imply that

$$\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big)=0.\tag{15}$$

Now taking the limit as $u \to \infty$ in the second summand of the right side of inequality (11), by condition $(\mathcal{C}2)$ we have

$$\begin{aligned}
\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big) &= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant \frac{1}{c}(Z_i+u)\}}\big)\\
&\leqslant \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant \frac{u}{c}\}}\big)=0.
\end{aligned}\tag{16}$$

We now see that the desired equality (10) follows from (11), (15), and (16). This means that all requirements of Lemma 1 hold for r.v.s $Z_i - c\theta_i$, $i \in \mathbb{N}$. $\square$

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